INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2007-08
Statistics - III, Midterm Examination, September 19, 2007

Marks are shown in parenthesis on the left of question numbers

Total Marks: 50 **Time: 2 hours, 30 minutes**

(5) 1. If $X_1, \ldots, X_n$ are independently distributed with $E(X_j) = j$ and $Var(X_j) = j^2$, $1 \leq j \leq n$, find the expected value of $Q = \sum_{j=1}^{n-1}(X_j - X_{j+1} + 1)^2$.

(7) 2. Let Z_1, Z_2, Z_3 be i.i.d. $N(0, \sigma^2)$. Find the conditional distribution of $Z_1^2 + Z_2^2 + Z_3^2$ given that $Z_1 + 2Z_2 + 3Z_3 = 0$.

(19) 3. Consider the model $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $\mathbf{X}_{n\times p}$ may not have full column rank.
(a) If $\hat{\beta}$ is the least squares estimator of β, show that $(\hat{\beta} - \beta)'\mathbf{X}'\mathbf{X}(\hat{\beta} - \beta)$ is distributed independently of the residual sum of squares.
(b) What is the probability distribution of $\hat{\beta}'\mathbf{X}'\mathbf{X}\hat{\beta}$?
(c) Find the maximum likelihood estimator of σ^2. Is it unbiased?
(d) Consider the case when $p = 2$, or there is only one regressor, X_1. When do we have independence of $\hat{\beta}_0$ and $\hat{\beta}_1$?

(5) 4. Let B^- be a generalized inverse of a symmetric matrix B and assume B^- is also symmetric. Show that if $P = BB^-$, then rank of B is the same as trace of P.

(14) 5. Consider the following model:

$y_1 = \theta + 2\phi + \epsilon_1$
$y_2 = \theta + \phi + \gamma + \epsilon_2$
$y_3 = \phi - \gamma + \epsilon_3$
$y_4 = 2\phi - 2\gamma + \epsilon_4,$

where θ, ϕ, γ are unknown constants and ϵ_i are uncorrelated random variables having mean 0 and variance σ^2.
(a) Show that $\gamma - \phi$ is estimable. What is its BLUE?
(b) Find the residual sum of squares. What is its degrees of freedom?
(c) Find an unbiased estimate of σ^2.